Self-consistent expansion for the Kardar-Parisi-Zhang equation with correlated noise

Eytan Katzav and Moshe Schwartz
Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

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A minor modification of the self-consistent expansion (SCE) for the Kardar-Parisi-Zhang (KPZ) system with uncorrelated noise is used to obtain the exponents in systems where the noise has spatial long-range correlations. For d-dimensional systems with correlations of the form $D(\vec{r}-\vec{r}\,',t-t')=2D_0|\vec{r}-\vec{r}\,'|^{2\rho-d}\delta(t-t')$, $(\rho>0)$, we find a lower critical dimension $d_0(\rho)=2+2\rho$, above which a perturbative Edwards-Wilkinson (EW) solution appears. Below the lower critical dimension two solutions exist, each in a different, distinct region of ρ. For small ρ's the solution of KPZ with uncorrelated noise is recovered. For large ρ's a ρ-dependent solution is found. The existence of only one solution in each region of ρ is not a result of a competition between two solutions but a direct outcome of the SCE equation. [S1063-651X(99)16310-4]

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The field of disorderly surface growth is today one of the most interesting and challenging fields in nonequilibrium statistical mechanics. Within this field, nonequilibrium roughening has received much attention. The first continuum equation used to study the growth of interfaces by particle deposition was the Edwards-Wilkinson model (EW) [1], which describes the dynamics of the interface by a noise driven diffusion equation. This model actually describes a process known as random deposition (RD) with surface relaxation, and forms a distinct universality class in growth phenomena. Before long, it was clear that an extension of this model was needed because of the nonlinear character of many deposition processes, such as ballistic deposition (BD), solid-on-solid deposition (SOS), and Eden growth. The first extension of the EW equation to include nonlinear terms was proposed by Kardar, Parisi, and Zhang [2], who suggested the addition of a nonlinear term proportional to the square of the height gradient.

$$\frac{\partial h}{\partial t}-\nu\nabla^2 h+g(\nabla h)^2=\eta(\vec{r},t), \tag{1}$$

where h is the height at $\vec{r}$ measured relative to its spatial average, and η is the fluctuation of the rate of deposition.

The KPZ equation is believed to belong to the same universality class as BD, SOS, and the Eden model—that is, in general dimension, different from the EW universality class. Although the KPZ equation cannot be solved due to its nonlinear character, the problem is exactly solvable in one dimension [3]. The exponents describing the roughness of the surface and the roughening process are known in two dimensions to a high accuracy from numerical simulations [4].

It follows, however, that in some experimental situations the measured scaling exponents are larger than the values predicted by KPZ [5,6]. A reason for that could be the uncorrelated nature of the noise assumed in the original KPZ model. In many systems, spatial correlations may exist, giving rise to scaling exponents different from those predicted by KPZ [7,8]. Numerical investigations, concerning discrete one-dimensional models with spatially correlated noise (BD [9,10,11], SOS [11,12], and direct integration of the KPZ equation [9]), confirmed that hypothesis. The noise we consider obeys

$$\langle \eta(\vec{r},t)\rangle=0, \tag{2}$$

and

$$\langle \eta(\vec{r},t)\,\eta(\vec{r}\,',t)\rangle=2D_0|\vec{r}-\vec{r}\,'|^{2\rho-d}\delta(t-t'), \tag{3}$$

where $d+1$ is the dimension of the system (d is the dimension of the surface).

Medina *et al.* [13] used dynamical renormalization-group (RG) analysis to study the KPZ equation with the above noise. One important result is that Galilean invariance is not destroyed by spatially correlated noise, so the scaling relation $\alpha+z=2$ remains valid (the scaling of $\langle[h(x,t)-h(0,0)]^2\rangle$ is given by $x^{2\alpha}f(t/x^z)$, where $z=\alpha/\beta$, $f(x)\sim x^{2\beta}$ for $x\ll 1$. z is called the dynamic exponent, α the roughness exponent, and β the growth exponent). As a result, there is only one independent exponent, and it is sufficient to give β:

$$\beta=\begin{cases}\dfrac{(d-2)^2}{12-8d-(d-2)^2} & 0<\rho\le\rho_0\\[2ex] \dfrac{(2\rho-d+2)}{(d+4-2\rho)} & \rho_0<\rho\le\rho_c\end{cases}, \tag{4}$$

where $\rho_0=d(d-2)/8(d-3/2)$ ($\rho_0=\frac{1}{4}$ for $d=1$), and $\rho_c=(d+1)/2$ ($\rho_c=1$ for $d=1$). It must be said that although the results of this analysis are at first sight formulated for any dimension, they give a sensible approximation for the scaling exponents only for $d=1$, where at least for $\rho=0$ the result is the exact result. For higher dimensions, however, even the results for $\rho=0$ display a discrepancy with the results of simulations. An interesting aspect of that calculation, however, is that for $\rho<\rho_0$, the long-range correlation is irrelevant. On the other hand, for $\rho>\rho_c$, higher order nonlinearities become relevant, hence the RG analysis collapses.

The one-dimensional predictions have been checked numerically. Some simulations have found good agreement with the exponents obtained in Ref. [13]. Hayot and Jayaprakash [14] found that these results extend even beyond the region of validity of the RG analysis, that is in simulations for $\rho>\rho_c$. Yet others, like Peng *et al.* [9] found systematic deviations from the predicted values.

Later theoretical efforts also led to the RG results. Halpin-Healy [15] investigated an equivalent problem (directed polymer in random media) using functional RG methods, finding complete agreement with the RG results of Medina *et al.*

Another line of research, which led to different scaling exponents, was taken by Zhang [16] who used a replica method to study the equivalent directed-polymer problem (DP) with correlated noise. He obtained in one dimension

$$\beta=\begin{cases}\dfrac{(1+2\rho)}{(3+2\rho)} & 0<\rho\leqslant\frac{1}{2}\\[2ex] \dfrac{(1+2\rho)}{(5-2\rho)} & \frac{1}{2}<\rho\leqslant 1.\end{cases} \tag{5}$$

Finally, Hentschel and Family [17] studied the scaling behavior for dissipative dynamical systems and proposed a new relation in one dimension,

$$\beta=\frac{1}{(3-2\rho)}\quad 0<\rho\leqslant\tfrac{1}{2}. \tag{6}$$

Peng *et al.* [9] compared numerical results for $d=1$ and theoretical predictions of RG analysis [13], to Zhang [16] and Hentschel and Family [17]. It turned out that the numerical results agree better with Hentschel and Family's prediction than with Zhang's prediction. Another interesting point was that for $\rho>\frac{1}{4}$, the numerical results agree very well with EW with correlated noise. This result might suggest that for sufficiently large ρ, the KPZ equation behaves like the linear theory even for $d=1$. Yet, this possibility is not consistent with any of the above-mentioned methods, which actually predict that the critical dimension (above which the KPZ equation behaves linearly) should be even higher than two for $\rho>0$, and not vice versa. Hence, no EW behavior should be seen for $d=1$.

Most of the work described above is concerned with a one-dimensional system where the exponents are exactly known for a very long time in the uncorrelated case $(\rho=0)$. The discrepancy among the various results at $d=1$ and finite ρ, and the fact that for $d>1$ the theoretical results, even for $\rho=0$, are obviously quite far from the numerical simulations, suggest that an independent approach should be used to try and clarify the issue.

In this paper we apply a method previously used by Schwartz and Edwards [18,19] concerning the KPZ equation. The advantages of that method are (i) it gives reasonable exponents for $\rho=0$ above one dimension; (ii) the modification needed to consider the correlated case instead of the uncorrelated one is minor and therefore the discussion extremely simple; and (iii) as will be seen later, in each region of ρ, there is a well-defined single solution.

The method is based on changing from the KPZ equation in Langevin form to a Fokker-Planck form and constructing a self-consistent expansion of the distribution of the field concerned.

The expansion is formulated in terms of ϕ_q and ω_q, where ϕ_q is the two-point function in momentum space, defined by $\phi_q=\langle h_q h_{-q}\rangle_S$, (the subscript S denotes steady state averaging), and ω_q is the characteristic frequency associated with h_q.

We expect that for small enough q, ϕ_q and ω_q are power laws in q,

$$\phi_q=Aq^{-\Gamma} \tag{7}$$

and

$$\omega_q=Bq^{\mu}. \tag{8}$$

[Since dynamic surface growth is a remarkably multidisciplinary field, there are almost as many notations as there are workers in the field. Therefore we give a brief translation of our notations to those most frequently used in this field:

$$\mu=z, \tag{9}$$

$$\alpha=\frac{\Gamma-d}{2}, \tag{10}$$

$$\beta=\frac{\alpha}{z}=\frac{\Gamma-d}{2\mu}. \tag{11}$$

The method produces, to second order in this expansion, two nonlinear coupled integral equations in ϕ_q and ω_q, that can be solved exactly in the asymptotic limit to yield the required scaling exponents governing the steady state behavior and the time evolution.

In fact, most of the discussion that appears in the previous paper [19] is general, and need not be revised. We are going to generalize the discussion of Schwartz and Edwards to a situation where the correlation function of the noise is given by Eqs. (2) and (3). This implies that in the consistency requirement for ϕ_q, D_0 is to be replaced by $D_0q^{-2\rho}$, resulting in

$$D_0q^{-2\rho}-\nu q^2\phi_q+I_1(q)\phi_q+I_2(q)=0. \tag{12}$$

The Herring consistency equation [20] for ω_q is

$$\omega_q-\nu q^2+J(q)=0. \tag{13}$$

(In fact, Herring's definition of ω_q is one of many possibilities, each leading to a different consistency equation. But it can be shown, as previously done in [19], that this does not affect the exponents (universality).)

The functions $I_1(q)$, $I_2(q)$, and $J(q)$ are given by

$$I_1(q)=\frac{2g^2}{(2\pi)^d}\int d^dl\,\frac{\vec{l}\cdot(\vec{q}-\vec{l})}{\omega_l+\omega_{q-l}+\omega_q}\times[\vec{l}\cdot\vec{q}\phi_l+(\vec{q}-\vec{l})\cdot\vec{q}\phi_{q-l}], \tag{14}$$

$$I_2(q)=\frac{2g^2}{(2\pi)^d}\int d^dl\,\frac{[\vec{l}\cdot(\vec{q}-\vec{l})]^2}{\omega_l+\omega_{q-l}+\omega_q}\phi_l\phi_{q-l}, \tag{15}$$

$$J(q)=\frac{2g^2}{(2\pi)^d}\int d^d l\,\frac{\vec{l}\cdot(\vec{q}-\vec{l})}{\omega_l+\omega_{q-l}}[\vec{l}\cdot\vec{q}\phi_l+(\vec{q}-\vec{l})\cdot\vec{q}\phi_{q-l}]. \tag{16}$$

Notice that we must require $d+2-\Gamma>0$ for the integral $J(q)$ to converge. The only modification of the uncorrelated equations is that D_0 is replaced by $D_0 q^{-2\rho}$.

We are interested in Eqs. (12) and (13) for small q's only. So we break up the integrals $I(q)$ and $J(q)$ into the sum of two contributions $I^>(q)$, $J^>(q)$ and $I^<(q)$, $J^<(q)$, corresponding to domains of $\vec{l}$ integration, with high and low momentum, respectively. We expand $I^>(q)$ and $J^>(q)$ for small q's and retain only the leading terms. Equations (12) and (13) reduce now to

$$D_0q^{-2\rho}+A_2-(\nu-A_1)q^2\phi_q+I_1^<(q)\phi_q+I_2^<(q)=0, \tag{17}$$

and

$$\omega_q-(\nu-A_3)q^2+J^<(q)=0. \tag{18}$$

At the mere price of renormalizing some constants in both equations, we are left with the integrals $I_1^<(q)$, $I_2^<(q)$, and $J^<(q)$ that can be calculated explicitly for small q's (just like in [19]), since for small $|\vec{l}|$'s, the power law form for ϕ_1 and ω_1 (or ϕ_{q-1} and ω_{q-1}) [Eqs. (7), (8)] can be used. The integrals $I_i^<(q)$ and $J^<(q)$ can now be evaluated

$$I_1^<(q),J^<(q)\,\alpha\begin{cases} q^2 & \text{for } d+2-\Gamma-\mu>0\,,\\ q^2\ln\dfrac{q_0}{q} & \text{for } d+2-\Gamma-\mu=0\,,\\ q^{d+4-\Gamma-\mu} & \text{for } d+2-\Gamma-\mu<0, \end{cases} \tag{19}$$

and

$$I_2^<(q)\,\alpha\begin{cases} \text{const} & \text{for } d+4-2\Gamma-\mu>0\,,\\ \text{const}\ln\dfrac{q_0}{q} & \text{for } d+4-2\Gamma-\mu=0\,,\\ q^{d+4-2\Gamma-\mu} & \text{for } d+4-2\Gamma-\mu<0, \end{cases} \tag{20}$$

where q_0 is the upper cutoff of the small $|\vec{l}|$ region.

We consider now the upper-right quadrant of the (Γ,μ) plane, where a solution may be expected. The lines $d+2-\Gamma-\mu=0$ and $d+4-2\Gamma-\mu=0$ divide the quadrant into four sectors. Next, we investigate each sector separately to decide whether a solution of Eqs. (17) and (18) can exist there or not (in the limit of small q's).

Sector α is defined by $d+2-\Gamma-\mu>0$ and $d+4-2\Gamma-\mu>0$. In this sector, Eqs. (17) and (18) reduce to

$$D_0q^{-2\rho}+A_2+A_2'-(\nu-A_1-A_1')Aq^{2-\Gamma}=0 \tag{21}$$

and

$$Bq^{\mu}-(\nu-A_3-A_3')q^2=0. \tag{22}$$

Notice that in general a new result can be obtained only for $\rho>0$, since for $\rho<0$ the term $D_0q^{-2\rho}$ is negligible compared to the second term (for small q's). The conclusion is $\mu=2$ and $-2\rho=2-\Gamma$. This result is the exact result in the case of the Edwards-Wilkinson model (linear model) with correlated noise. By definition of the sector it follows that this can happen only for $d>2+4\rho$.

Sector β is defined by $d+2-\Gamma-\mu>0$ and $d+4-2\Gamma-\mu<0$. In this sector, Eq. (21) is replaced by

$$D_0q^{-2\rho}+A_2-(\nu-A_1-A_1')Aq^{2-\Gamma}+C_2q^{d+4-2\Gamma-\mu}=0. \tag{23}$$

The last term on the left-hand side of this equation is negligible compared to the third term, due to the defining condition $d+2-\Gamma-\mu>0$. It also appears from the two defining conditions that $2-\Gamma<0$, so that the second term is also negligible compared to the third term. Therefore, a possible new solution in this sector appears when $-2\rho=2-\Gamma$, which is again just the EW solution. Since Eq. (22) is unchanged, here also $\mu=2$, which implies that this solution exists only if $2+2\rho<d<2+4\rho$.

Combining the results for sectors α and β, we see that the EW solution is possible only for $d>2+2\rho$. Therefore, the lower critical dimension is $d_c=2+2\rho$.

Sector γ is defined by $d+2-\Gamma-\mu<0$ and $d+4-2\Gamma-\mu>0$. In this sector, Eq. (21) is replaced by

$$D_0q^{-2\rho}+A_2-(\nu-A_1)Aq^{2-\Gamma}+C'q^{d+4-2\Gamma-\mu}=0. \tag{24}$$

The two equations defining the sector imply that the two last terms on the left-hand side of the equation are negligible compared with the constant. So in order to get a solution we must have $\rho=0$, and $D_0+A_2=0$, which is impossible, because D_0 and A_2 are both positive definite, and anyhow we are dealing with $\rho>0$.

Sector δ is defined by $d+2-\Gamma-\mu<0$ and $d+4-2\Gamma-\mu<0$. In this sector, Eqs. (17) and (18) take the form

$$D_0q^{-2\rho}+A_2-(\nu-A_1)Aq^{2-\Gamma}+\frac{2g^2}{(2\pi)^d}\frac{A^2}{B}F(\Gamma,\mu)q^{d+4-2\Gamma-\mu}=0, \tag{25}$$

and

$$Bq^{\mu}-(\nu-A_3)q^2+\frac{2g^2}{(2\pi)^d}\frac{A}{B}G(\Gamma,\mu)q^{d+4-\Gamma-\mu}=0, \tag{26}$$

where $F(\Gamma,\mu)$ is given by

$$F(\Gamma,\mu)=-\int d^dt\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]}[\vec{t}\cdot\hat{e}\cdot t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}\cdot|\hat{e}-\vec{t}|^{-\Gamma}]+\int d^dt\,\frac{[\vec{t}\cdot(\hat{e}-\vec{t})]^2}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}+1]}\cdot t^{-\Gamma}\cdot|\hat{e}-\vec{t}|^{-\Gamma}, \tag{27}$$

and $G(\Gamma,\mu)$ by the following:

$$G(\Gamma,\mu)=\int d^d t\,\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{[t^{\mu}+|\hat{e}-\vec{t}|^{\mu}]}[\vec{t}\cdot\hat{e}\cdot t^{-\Gamma}+(\hat{e}-\vec{t})\cdot\hat{e}\cdot|\hat{e}-\vec{t}|^{-\Gamma}]. \tag{28}$$

$\hat{e}$ is a unit vector in an arbitrary direction, and the $\vec{t}$ integration is over all d-dimensional space.

Equation (26) leads, just as in the original paper [19], to the scaling relation $d+4-\Gamma-2\mu=0$. The defining conditions $d+2-\Gamma-\mu<0$ and $d+4-2\Gamma-\mu<0$ allows us to neglect the second and the third terms in Eq. (25) compared to the last term.

We are now facing two possible solutions. Either the last term on the left-hand side of the equation dominates over the first term, or they have the same power of q. In the first case, the exponent Γ will be given by the solution of the equation $F[\Gamma,\mu(\Gamma)]=0$, where $\mu(\Gamma)$ is related to Γ by the scaling relation $\mu=(d+4-\Gamma)/2$. The exponents are thus exactly the Γ_0 and μ_0 obtained for the uncorrelated case. (Note that Γ_0 and μ_0 are the solutions of the transcendental equation $F[\Gamma,\mu(\Gamma)]=0$. For example, in one dimension it can be shown analytically that $\Gamma_0=2$, and in two dimensions a numerical solution of the equation yields $\Gamma_0=2.59$.) We must remember, however, that such a solution is obtained by requiring that the last term on the left-hand side dominates over the first one. This yields a necessary condition for the existence of such a solution, $\rho<\rho_0(d)=[3\Gamma_0(d)-d-4]/4$. For $\rho>\rho_0(d)$, if a solution exists at all, it must be the new solution $\Gamma=(d+4+4\rho)/3$. A necessary condition that allows such a solution is that the coefficient of $q^{-2\rho}$ in Eq. (25) vanishes, but that is possible only if $F[\Gamma,\mu(\Gamma)]<0$. The function $F[\Gamma,\mu(\Gamma)]$ changes sign at $\Gamma=\Gamma_0(d)$, and it turns out that for $\Gamma>\Gamma_0(d)$, it becomes negative. This implies at once that for $\rho<\rho_0(d)$, the only solution is $\Gamma=\Gamma_0(d)$. Namely, the long-range correlation is not relevant. For $\rho>\rho_0(d)$, the new solution, $\Gamma=(d+4+4\rho)/3$, exists, and it is the only one in that region of ρ.

Note that the applicability of the method is limited to $\rho<(d+1)/2$, since Γ must be less than $d+2$ in order to have $J(q)$ convergent.

We now give a translation of these results to the frequently used notation:

$$\beta=\begin{cases}\dfrac{\Gamma_0-d}{d+4-\Gamma_0} & 0<\rho\leqslant\max\left\{0,\dfrac{d}{2}-1,\rho_0\right\}\\[2ex] \dfrac{(2\rho-d+2)}{(d+4-2\rho)} & \max\left\{0,\dfrac{d}{2}-1,\rho_0\right\}<\rho\leqslant\rho_c\end{cases}, \tag{29}$$

where $\rho_0=(3\Gamma_0-d-4)/4$, $\rho_c=(d+1)/2$, and Γ_0 is the exponent obtained in the uncorrelated case (which is the solution of the transcendental equation $F[\Gamma,\mu(\Gamma)]=0$).

In this paper we presented a straightforward generalization of a self-consistent expansion for the KPZ equation to include correlated noise characterized by $D(\vec{r}-\vec{r}\,')=D_0|\vec{r}-\vec{r}\,'|^{2\rho-d}$. In one dimension we recover the result of Medina *et al.* [13]. For $d>1$ we still find that for small enough ρ the long-range correlations are irrelevant but the actual results are quite different, reducing at $\rho=0$ to a much more sensible result. For example, for $d=2$ the RG result is $\beta=0$ [13], numerical results suggest β between 0.2 and 0.25 [4], and our result is $\beta=0.17$.

We also find that above $d_c(\rho)=2+2\rho$, a weak coupling EW solution becomes possible. These results are still in disagreement with Peng *et al.* [9], who find in one dimension an EW behavior for large ρ, but are in agreement with the smaller ρ's of Hentschel and Family [17]. This discrepancy has to be clarified in the future. At present, we feel that the asymptotic regime probably has not been reached due to the parameters used in the numerical simulations of the KPZ equation.

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